CONVECTIVE HEAT TRANSFER

Mohammad Goharkhah
Department of Mechanical Engineering, Sahand University of Technology,
Tabriz, Iran
CHAPTER 5

NATURAL CONVECTION HEAT TRANSFER
BASIC CONCEPTS
The air layer adjacent to the wall expands (becomes lighter, less dense) and rises. At the same time, the cold reservoir fluid is displaced downward. Thus, the wall-reservoir temperature difference induces the ‘natural circulation’

heating → expansion → cooling → compression
Important notes

- The loop-shaped flow is the succession of four processes, heating → expansion → cooling → compression.
- The imaginary heat engine cycle should be capable of delivering work if we insert a suitably designed propeller in the stream.
- In the absence of work-collecting devices (e.g., windmill wheels), the heat engine cycle drives its working fluid fast enough so that its entire work output is destroyed because of irreversibilities due to friction between adjacent fluid layers and heat transfer along finite temperature gradients.
- The fundamental difference between forced convection and natural convection is that, in forced convection, the engine that drives the flow is external, whereas in natural convection the engine is built into the flow itself.
- Mathematically, the fluid flow problem in forced convection is decoupled from the heat transfer problem.
Objective

To predict the heat transfer rate $Q$ \textit{when} the wall–reservoir temperature difference is known,

$$Q = (HW)h_{0-H}(T_0 - T_{\infty})$$

\textit{The wall area}

In other words, the objective is to calculate the wall-averaged heat transfer coefficient $h_{0-H}$.

Complete Navier–Stokes equations for the steady constant-property two-dim. flow

\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\
\rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) &= -\frac{\partial P}{\partial x} + \mu \nabla^2 u \\
\rho \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) &= -\frac{\partial P}{\partial y} + \mu \nabla^2 v - \rho g \\
u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} &= \alpha \nabla^2 T
\end{align*}
LAMINAR BOUNDARY LAYER EQUATIONS

- The N. S. Equations reduce to simpler forms if the focus is on the boundary layer region
  \( (x \sim \delta_T, y \sim H, \text{and } \delta_T << H) \).

- Thus, only the \( \frac{\partial^2}{\partial x^2} \) term survives in the \( \nabla^2 \) operator.

- The transversal momentum equation reduces to the statement that in the boundary layer, the pressure is a function of longitudinal position only,
  \[ \frac{\partial P}{\partial y} = \frac{dP}{dy} = \frac{dP_\infty}{dy} \]

The boundary layer equations for momentum and energy

\[
\rho \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{dP_\infty}{dy} + \mu \frac{\partial^2 v}{\partial x^2} - \rho g \\
u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial x^2}
\]
LAMINAR BOUNDARY LAYER EQUATIONS

\[ \rho \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{dP_\infty}{dy} + \mu \frac{\partial^2 v}{\partial x^2} - \rho g \]

The hydrostatic pressure gradient dictated by the reservoir fluid of density \( \rho_\infty \), \( \frac{dP_\infty}{dy} = -\rho_\infty g \)

\[ u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial x^2} \]

\[ \rho \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = \mu \frac{\partial^2 v}{\partial x^2} + (\rho_\infty - \rho) g \]

Note that:

- The flow is driven by the density field \( \rho(x, y) \) generated by the temperature field \( T(x, y) \).
- Energy and momentum equations are coupled via the equation of state of the fluid; for example, if the fluid behaves according to the ideal gas model \( P = \rho RT \)

\[ P = \rho RT \Rightarrow \rho = \frac{P_\infty / R}{T} \quad \text{and} \quad \rho_\infty = \frac{P_\infty / R}{T_\infty} \Rightarrow \rho - \rho_\infty = \rho \left( 1 - \frac{T}{T_\infty} \right) \]

\[ \rho_\infty - \rho \left( 1 - \frac{\rho_\infty - \rho}{\rho_\infty} \right)^{-1} = \frac{T - T_\infty}{T_\infty} \Rightarrow \rho \simeq \rho_\infty \left[ 1 - \frac{1}{T_\infty} (T - T_\infty) + \cdots \right] \]

in the limit \( (T - T_\infty) \ll T_\infty \)
LAMINAR BOUNDARY LAYER EQUATIONS

\[ \rho \approx \rho_\infty \left[ 1 - \frac{1}{T_\infty} (T - T_\infty) + \cdots \right] \]

In general, for fluids that are not necessarily ideal gases, this expression is written as

\[ \rho \approx \rho_\infty \left[ 1 - \beta (T - T_\infty) + \cdots \right] \]

\[ \beta = -\frac{1}{\rho} \left( \frac{\partial \rho}{\partial T} \right)_p \]

this dimensionless product is considerably smaller than unity

The Boussinesq-approximated momentum equation

\[ u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = v \frac{\partial^2 v}{\partial x^2} + g \beta (T - T_\infty) \]

Impermeable, no slip, isothermal wall: \( u = v = 0 \) and \( T = T_0 \) at \( x = 0 \)

Stagnant, isothermal infinite reservoir: \( v = 0 \) and \( T = T_\infty \) as \( x \to \infty \)
Consider the conservation of mass, momentum, and energy in the \textit{thermal} boundary layer region \((x \sim \delta_T, y \sim H)\)

\[
\frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial x^2}
\]

\[
\frac{\Delta T}{\delta_T}, \quad \frac{\Delta T}{H} \sim \frac{\Delta T}{\delta_T^2} \quad \text{Convection} \quad \alpha \frac{\Delta T}{\delta_T^2} \quad \text{Conduction}
\]

\[
\Delta T = T_0 - T_\infty
\]

\[
v \sim \frac{\alpha H}{\delta_T^2}
\]

\[
v \frac{\Delta T}{H} \sim \alpha \frac{\Delta T}{\delta_T^2}
\]

Two convection terms are of order \((v \Delta T)/H\)

\[
\frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = v \frac{\partial^2 v}{\partial x^2} + g\beta(T - T_\infty)
\]

\[
\frac{u}{\delta_T}, \quad \frac{v}{H} \quad \frac{v v}{\delta_T^2} \quad g\beta \Delta T
\]

Inertia

Friction

Buoyancy

The two inertia terms are of order \(v^2/H\)

\[
\frac{u}{\delta_T} \sim \frac{v}{H}
\]

mass conservation
Under what conditions the $\delta_T$ layer is ruled by the inertia $\sim$ buoyancy balance, as opposed to the friction $\sim$ buoyancy balance.

- The competition between inertia and friction is decided by a fluid property, the Prandtl number.

- High-Pr fluids will form a $\delta_T$ layer ruled by the friction–buoyancy balance, while low-Pr fluids will form a $\delta_T$ layer with buoyancy balanced by inertia.
SCALE ANALYSIS- High-Pr Fluids

\[
\left( \frac{H}{\delta_T} \right)^4 \text{Ra}_H^{-1} \text{Pr}^{-1} \quad \text{Inertia} \quad \left( \frac{H}{\delta_T} \right)^4 \text{Ra}_H^{-1} \quad \text{Friction} \quad 1 \quad \text{Buoyancy} \quad \text{Pr}>> 1 \quad \text{The friction–buoyancy balance} \quad \delta_T \sim H \text{Ra}_H^{-1/4} \quad v \sim \frac{\alpha H}{\delta_T^2}
\]

- the \( \delta_T \)-thick layer effects the transition from \( T_0 \) to \( T_\infty \) and at the same time drives fluid upward.
- The fluid motion is not restricted to a layer of thickness \( \delta_T \). It is possible for the heated \( \delta_T \) layer to entrain viscously a layer of outer (unheated) fluid with thickness \( \delta \).
- Since the outer fluid is isothermal, the buoyancy effect is absent, the conservation of momentum implies that there is an inertia ~ friction balance in a layer of thickness \( \delta \):

\[
\frac{v}{H} \sim \frac{v}{\delta^2} \quad \Rightarrow \delta \sim H \text{Ra}_H^{-1/4} \text{Pr}^{1/2} \quad \Rightarrow \frac{\delta}{\delta_T} \sim \text{Pr}^{1/2} > 1
\]
The velocity profile is described by two length scales ($\delta_T$ and $\delta$), not by a single length scale ($\delta$) as in forced convection. The velocity scale is reached within a thin layer $\delta_T$, while the velocity decays to zero within a thick layer $\delta$. The ratio $\frac{\delta}{\delta_T} \sim Pr^{1/2} > 1$. 
SCALE ANALYSIS - low-Pr Fluids

\[
\begin{align*}
\left( \frac{H}{\delta_T} \right)^4 \text{Ra}_H & \sim \frac{1}{\text{Pr}} \\
\left( \frac{H}{\delta_T} \right)^4 \text{Ra}_H^{-1} & \sim \text{Friction} \\
\left( \frac{H}{\delta_T} \right)^4 \text{Ra}_H^{-1} & \sim \text{Buoyancy} \\
\end{align*}
\]

\[
\begin{align*}
\text{Pr} & \ll 1 \\
\delta_T & \sim H (\text{Ra}_H \text{ Pr})^{-1/4} \\
\nu & \sim \frac{\alpha}{H} (\text{Ra}_H \text{ Pr})^{1/2} \\
\text{Nu} & = \frac{hH}{k} \sim (\text{Ra}_H \text{ Pr})^{1/4} \\
\text{Bo}_H & = \text{Ra}_H \text{ Pr} = \frac{g \beta \Delta T H^3}{\alpha^2}
\end{align*}
\]

Boussinesq number
The $\delta_T$ layer is driven upward by buoyancy and restrained by inertia.

Outside the $\delta_T$ layer, where the fluid is isothermal and the buoyancy effect is absent, the fluid is motionless. The velocity profile must then be as wide as the temperature profile. However, since the no-slip condition still applies at the wall, the location of the velocity peak is an important second-length scale in the description of the velocity profile.

$\delta_v$ is the thickness of a very thin layer right near the wall, a layer in which the buoyancy-driven fluid is restrained viscously by the wall.

**Equations:**

\[
\begin{align*}
\text{buoyancy \sim friction balance} \\
\frac{v}{\delta_v^2} & \sim g \beta \Delta T \\
v & \sim \frac{\alpha}{H} (\text{Ra}_H \ Pr)^{1/2}
\end{align*}
\]

\[
\begin{align*}
\delta_v & \sim H \ Gr_H^{-1/4} \\
Gr_H & = \frac{g \beta \Delta T H^3}{v^2} = \frac{\text{Ra}_H}{\text{Pr}}
\end{align*}
\]

\[\Rightarrow \frac{\delta_v}{\delta_T} \sim \text{Pr}^{1/2} < 1\]
Summary of flow and heat transfer scales in a natural convection boundary layer along a vertical wall

<table>
<thead>
<tr>
<th>Prandtl Number Range</th>
<th>Thermal Boundary Layer Thickness</th>
<th>Wall Jet Velocity Profile</th>
<th>Nusselt Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pr &gt; 1</td>
<td>$H \text{Ra}_H^{-1/4}$</td>
<td>$H \text{Ra}_H^{-1/4}$</td>
<td>$\frac{\alpha}{H} \text{Ra}_H^{1/2}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\text{Pr}^{1/2}(H \text{Ra}_H^{-1/4})$</td>
<td>$\text{Ra}_H^{1/4}$</td>
</tr>
<tr>
<td>Pr &lt; 1</td>
<td>$\text{Pr}^{-1/4}(H \text{Ra}_H^{-1/4})$</td>
<td>$\text{Pr}^{1/4}(H \text{Ra}_H^{-1/4})$</td>
<td>$\text{Pr}^{1/4}(H \text{Ra}_H^{-1/4})$</td>
</tr>
</tbody>
</table>

\[ \text{Nu} = \frac{hH}{k} \]
Dimensionless numbers

The important groups in external natural convection are the Rayleigh number for $Pr > 1$ fluids and the Boussinesq number for $Pr < 1$ fluids.

\[
\begin{align*}
\text{Ra}_H^{1/4} & \sim \frac{\text{wall height}}{\text{thermal boundary layer thickness}} \quad (Pr > 1) \\
\text{Bo}_H^{1/4} & \sim \frac{\text{wall height}}{\text{thermal boundary layer thickness}} \quad (Pr < 1) \\
\text{Gr}_H^{1/4} & \sim \frac{\text{wall height}}{\text{wall shear layer thickness}} \quad (Pr < 1)
\end{align*}
\]
INTEGRAL SOLUTION

Objective To determine the actual $y$ variation of features such as local heat flux ($q$), thermal boundary layer thickness ($\delta T$), and wall jet velocity profiles.

Integrate the momentum equation and the energy equation from the wall ($x = 0$) to a far enough plane $x = X$ in the motionless isothermal cold reservoir

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = v \frac{\partial^2 v}{\partial x^2} + g \beta (T - T_\infty)$$  momentum equation

$$\frac{\partial}{\partial x} (uv) + \frac{\partial}{\partial y} (v^2) = v \frac{\partial^2 v}{\partial x^2} + g \beta (T - T_\infty)$$

$$u_x \frac{v}{v_0} - u_0 \frac{v_0}{v_0} + \frac{d}{dy} \int_0^X v^2 \, dx = v \frac{\partial v}{\partial x} \bigg|_0^X - v \frac{\partial v}{\partial x} \bigg|_0 + g \beta \int_0^X (T - T_\infty) \, dx$$

$$\frac{d}{dy} \int_0^X v^2 \, dx = - v \frac{\partial v}{\partial x} \bigg|_{x=0} + g \beta \int_0^X (T - T_\infty) \, dx$$
INTEGRAL SOLUTION

\[ \frac{\partial}{\partial x} (uT) + \frac{\partial}{\partial y} (vT) = \alpha \frac{\partial^2 T}{\partial x^2} \]

energy equation

\[ u_x \frac{T_x}{T_\infty} - \frac{u_0}{0} T_0 + \frac{d}{dy} \int_0^X vT \, dx = \alpha \left( \frac{\partial T}{\partial x} \right)_x - \alpha \left( \frac{\partial T}{\partial x} \right)_0 \]

where \( u_x \) is the so-called "entrainment" velocity; we find it by integrating the mass continuity equation,

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \Rightarrow \quad u_x - \frac{u_0}{0} + \frac{d}{dy} \int_0^X v \, dx = 0 \]

Back in the energy equation, we now have

\[ -T_\infty \frac{d}{dy} \int_0^X v \, dx + \frac{d}{dy} \int_0^X vT \, dx = -\alpha \left( \frac{\partial T}{\partial x} \right)_0 \]

\[ \frac{d}{dy} \int_0^X v(T_\infty - T) \, dx = \alpha \left( \frac{\partial T}{\partial x} \right)_{x=0} \]
We must carry out the integral analysis in two parts, for $\text{Pr} > 1$ and $\text{Pr} < 1$, as the boundary layer constitution changes dramatically across $\text{Pr} \sim 1$.

the velocity profile shape is governed by two length scales, one for the wall shear layer and another for the overall thickness of the moving layer of fluid.
INTEGRAL SOLUTION- High-Pr Fluids

\[ T - T_\infty = \Delta T e^{-x/\delta_T} \quad V, \delta_T, \text{and } \delta \text{ are unknown functions of altitude } (y) \]
\[ v = Ve^{-x/\delta} (1 - e^{-x/\delta_T}) \]

Substituting the profiles into the momentum and energy integrals and setting \( X \to \infty \) yield

\[
\begin{align*}
\frac{d}{dy} \left[ \frac{V^2 \delta q^2}{2 (2 + q) (1 + q)} \right] &= -\frac{\nu V q}{\delta} + g\beta \Delta T \frac{\delta}{q} \\
\frac{d}{dy} \left[ \frac{V \delta}{(1 + q) (1 + 2q)} \right] &= \frac{\alpha}{\delta} \\
q(Pr) &= \frac{\delta}{\delta_T}
\end{align*}
\]

We have two equations for three unknowns: \( V(y), \delta(y), \text{and } q(Pr) \). Older integral analyses avoided this problem by assuming that \( \delta T = \delta \). However, since a great deal of the information relating to boundary layer geometry is buried in the \( \delta/\delta T \) function, it is instructive to meet the challenge of an integral analysis with \( \delta \neq \delta T \).

We can select as a third equation a force balance: One that is both clear and analytically brief is the statement that in the no-slip layer \( 0 < x < 0^+ \), the inertia terms are zero:

\[ 0 = \nu \frac{\partial^2 v}{\partial x^2} + g\beta (T_0 - T_\infty) \]
The equations are then solved for $V$, $\delta$, and $q$ by first noting that $\delta \sim y^{1/4}$ and $V \sim y^{1/2}$.

$$\Pr = \frac{5}{6} q^2 \left( \frac{q + \frac{1}{2}}{q + 2} \right) \quad \text{Nu} = \frac{q''}{T_0 - T_\infty} \frac{y}{k} = \left[ \frac{3}{8} \frac{q^3}{(q + 1) (q + \frac{1}{2}) (q + 2)} \right]^{1/4} \text{Ra}_y^{1/4}$$

In the limit $\Pr \to \infty$, this solution reduces to

$$\frac{\delta}{\delta_T} = \left( \frac{6}{5} \Pr \right)^{1/2} \quad \text{and} \quad \text{Nu} = 0.783 \text{Ra}_y^{1/4}$$
$T - T_\infty = \Delta T e^{-x/\delta_T}$

$v = V_1 e^{-x/\delta_T} (1 - e^{-x/\delta_v})$

Again using the integral equations and noticing from the table that $\delta_T \sim y^{1/4}$, $\delta_v \sim y^{1/4}$, and $V1 \sim y^{1/2}$, the solution reduces to

$$\text{Pr} = \frac{5}{3} \left( \frac{q_1}{1 + q_1} \right)^2, \quad q_1 = \frac{\delta_v}{\delta_T}$$

$$\text{Nu} = \frac{q''}{T_0 - T_\infty} \frac{y}{k} = \left( \frac{3}{8} \right)^{1/4} \left( \frac{q_1}{2q_1 + 1} \right)^{1/2} \text{Ra}_y^{1/4}$$

In the limit $\text{Pr} \to 0$, these results become

$$\frac{\delta_v}{\delta_T} = \left( \frac{3}{5} \text{Pr} \right)^{1/2} \quad \text{Nu} = 0.689 \left( \text{Pr} \text{ Ra}_y \right)^{1/4}$$
Example

Perform an integral analysis of the natural convection boundary layer by assuming the following temperature and velocity profiles:

\[ T - T_\infty = \Delta T \left( 1 - \frac{x}{\delta_T} \right)^2, \quad v = V_x \frac{x}{\delta} \left( 1 - \frac{x}{\delta} \right)^2 \]

For simplicity, assume that \( \delta = \delta T \). Show that the local Nusselt number is given by

\[ \text{Nu} = \frac{q''}{T_0 - T_\infty} \frac{y}{k} = 0.508 \left( 1 + \frac{20}{(21) \text{Pr}} \right)^{-1/4} \text{Ra}_y^{1/4} \]
$T - T_\infty = \Delta T \left(1 - \frac{x}{\delta}\right)^2$ and $v = V \frac{x}{\delta} \left(1 - \frac{x}{\delta}\right)^2$

We work first on the momentum equation:

$$\frac{d}{dy} \int_0^X v^2 \, dx = -v \left(\frac{\partial v}{\partial x}\right)_0 + g\beta \int_0^X (T - T_\infty) \, dx$$

$$v^2 \delta \int_0^1 m^2 (1 - m)^4 \, dm \quad \frac{V}{\delta} \quad \Delta T \delta \int_0^1 (1 - m)^2 \, dm$$

$$\frac{1}{105} \quad \frac{1}{3}$$

$$\Rightarrow \frac{d}{dy} \left(\frac{V^2 \delta}{105}\right) = -v \frac{V}{\delta} + \frac{1}{3} \Delta T \delta g\beta$$

Turning now our attention to the energy equation, we have

$$\frac{d}{dy} \int_0^X v (T_\infty - T) \, dx = \alpha \left(\frac{\partial T}{\partial x}\right)_0$$

$$- V \Delta T \delta \int_0^1 m(1 - m)^4 \, dx \quad - \frac{2 \Delta T}{\delta}$$

$$\frac{1}{30}$$

$$\Rightarrow \frac{d}{dy} (V \delta) = \frac{60}{\delta}$$
At this stage we have two equations, \( A \) and \( B \) for two unknowns, \( V(y) \) and \( \delta(y) \). The \( y \)-dependence of \( V \) and \( \delta \) is already known from scale analysis

\[
V = C_V y^{1/2}, \quad \delta = C_\delta y^{1/4}
\]

\[
C_V = 5.17 v \left( \Pr + \frac{20}{21} \right)^{-1/2} \left( \frac{g \beta \Delta T}{v^2} \right)^{1/2}
\]

\[
C_\delta = 3.93 \Pr^{-1/2} \left( \Pr + \frac{20}{21} \right)^{1/4} \left( \frac{g \beta \Delta T}{v^2} \right)^{-1/4}
\]

\[
\frac{\delta}{y} = 3.93 \left( \frac{20/21}{\Pr} + 1 \right)^{1/4} \text{Ra}_y^{-1/4}
\]

\[
\text{Nu} = \frac{q''}{\Delta T} \frac{y}{k} = \frac{-k (\partial T/\partial x)_0}{\Delta T} \frac{y}{k} = 2 \frac{y}{\delta} = 0.508 \left( \frac{20/21}{\Pr} + 1 \right)^{-1/4} \text{Ra}_y^{1/4}
\]

The above is Squire’s result
SIMILARITY SOLUTION

From below table and the integral solution, we know that any length scale of the boundary layer region is proportional to \( y^{1/4} \). The dimensionless similarity variable \( \eta(x, y) \) can then be constructed as \( x \) divided by any of the length scales summarized in the table; selecting the \( \text{Pr} > 1 \) thermal boundary layer thickness \( y \, \text{Ra}_y^{-1/4} \) as the most appropriate length scale, the similarity variable emerges as

\[
\eta = \frac{x}{y} \, \text{Ra}_y^{1/4}
\]

**Table 4.1  Summary of flow and heat transfer scales in a natural convection boundary layer along a vertical wall**

<table>
<thead>
<tr>
<th>Prandtl Number</th>
<th>Thermal Boundary Layer Thickness</th>
<th>Wall Jet Velocity Profile</th>
<th>Nusselt Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pr &gt; 1</td>
<td>( H , \text{Ra}_H^{-1/4} )</td>
<td>( H , \text{Ra}_H^{-1/4} )</td>
<td>( \frac{\text{Pr}^{1/2} (H , \text{Ra}_H^{-1/4})}{H} \frac{\alpha}{H} , \text{Ra}_H^{1/2} )</td>
</tr>
<tr>
<td>Pr &lt; 1</td>
<td>( \text{Pr}^{-1/4} (H , \text{Ra}_H^{-1/4}) )</td>
<td>( \text{Pr}^{1/4} (H , \text{Ra}_H^{-1/4}) )</td>
<td>( \text{Pr}^{-1/4} (H , \text{Ra}_H^{-1/4}) )</td>
</tr>
</tbody>
</table>
SIMILARITY SOLUTION

\[ u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \]

Stream function

Momentum and Energy Eqs.

\[
\begin{align*}
\frac{\partial \psi}{\partial y} \frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial T}{\partial y} &= \alpha \frac{\partial^2 T}{\partial x^2} \\
-\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial x \partial y} &= -\nu \frac{\partial^3 \psi}{\partial x^3} + g\beta (T - T_\infty)
\end{align*}
\]

For the vertical velocity profile \( v \), from the fourth column of the table where \( Pr > 1 \), we select the expression

\[ v = \frac{\alpha}{y} \text{Ra}_y^{1/2} G(\eta, \text{Pr}) \]

the dimensionless similarity profile of the wall jet

\[ \frac{T - T_\infty}{T_0 - T_\infty} = \theta(\eta, \text{Pr}) \]

the dimensionless temperature profile

\[ v = -\frac{\partial \psi}{\partial x} \quad \psi = \alpha \text{Ra}_y^{1/4} F(\eta, \text{Pr}) \]

\[ G = -\frac{\partial F}{\partial \eta} \]
SIMILARITY SOLUTION

\( \frac{3}{4} F' \theta' = \theta'' \)

\[ \frac{1}{\text{Pr}} \left( \frac{1}{2} F'^2 - \frac{3}{4} F F'' \right) = -F''' + \theta \]

(i) At \( x = 0 \), \( u = 0 \) \( F = 0 \)

\( \eta = 0 \) \( v = 0 \) \( F' = 0 \)

\( T = T_0 \) \( \theta = 1 \)

(ii) As \( x \to \infty \), \( v = 0 \) \( F' = 0 \)

\( \eta \to \infty \) \( T = T_\infty \) \( \theta = 0 \)
SIMILARITY SOLUTION

\[ \eta = \frac{x}{y} \, Ra^{1/4} \]

(a)
SIMILARITY SOLUTION

\[ G = \sqrt{\frac{\gamma}{y}} Ra^{-1/2} \]

\[ \eta = \frac{x}{y} Ra^{1/4} \]

(b)
SIMILARITY SOLUTION

\[ \text{Nu} = \frac{h_y}{k} = -\left(\theta'\right)_{\eta=0} \text{Ra}_y^{1/4} \]

... a function of the Prandtl number

Table 4.2 Similarity solution heat transfer results for natural convection boundary layer along a vertical isothermal wall

<table>
<thead>
<tr>
<th>Pr</th>
<th>0.01</th>
<th>0.72</th>
<th>1</th>
<th>2</th>
<th>10</th>
<th>100</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nu Ra_y^{-1/4}</td>
<td>0.162</td>
<td>0.387</td>
<td>0.401</td>
<td>0.426</td>
<td>0.465</td>
<td>0.490</td>
<td>0.499</td>
</tr>
</tbody>
</table>

\[ \text{Nu} = 0.503 \text{Ra}_y^{1/4} \quad \text{as Pr} \to \infty \]

\[ \text{Nu} = 0.6(\text{Ra}_y \text{ Pr})^{1/4} \quad \text{as Pr} \to 0 \]

\[ h \sim y^{-1/4} \quad h_{0-H} = (4/3)h(y = H) \]

\[ \text{Nu}_{0-H} = 0.671 \text{Ra}_H^{1/4} \quad \text{as Pr} \to \infty \]

\[ \text{Nu}_{0-H} = 0.8(\text{Ra}_H \text{ Pr})^{1/4} \quad \text{as Pr} \to 0 \]
Isothermal b. c. VS Uniform Heat Flux b. c.

(a) Isothermal wall

\[ \delta_T \sim y^{1/4} \]

\[ \Delta T \]

\[ q'' \sim y^{-1/4} \]

(b) Uniform wall heat flux

\[ \delta_T \sim y^{1/5} \]

\[ \Delta T \sim y^{1/5} \]

\[ q'' \]

Flow

identical \( T \) and \( \delta_T \) functions of \( y \).

both \( \Delta T \) and the product \( q \delta_T \) are independent of \( y \).
UNIFORM WALL HEAT FLUX

Objective
To predict the wall—ambient temperature difference $T_0(y) - T_\infty$ when the uniform heat flux $q$ is given.

$$\Pr \gg 1 \quad \delta_T \sim H \, Ra_H^{-1/4} \quad \delta_T \sim H \left( \frac{g \beta \Delta T H^3}{\alpha \nu} \right)^{-1/4}$$

$$q'' \sim k \frac{\Delta T}{\delta_T} \quad \Rightarrow \quad \delta_T \sim H \, Ra_\ast_H^{-1/5}$$

$Ra_\ast$ is a Rayleigh number based on heat flux $q''$

$$Ra_\ast = \frac{g \beta H^4 q''}{\alpha \nu k}$$

Note that both $\delta T$ and $\Delta T$ are proportional to $H^{1/5}$; because the $H$-averaged quantities are proportional to $H^{1/5}$, the local values of $\delta T$ and $T$ are proportional to $y^{1/5}$.

$$Nu = \frac{q''}{T_0(y) - T_\infty} \frac{y}{k} \quad \Rightarrow \quad Nu \sim \frac{H}{\delta_T} \sim Ra_\ast_H^{1/5}$$
UNIFORM WALL HEAT FLUX

low-Prandtl number fluids, \( \delta_T \sim H (Ra_H \ Pr)^{-1/4} \)

\[ \delta_T \sim H (Ra*_{H} \ Pr)^{-1/5} \]
\[ \Delta T \sim \frac{q''}{k} H (Ra*_{H} \ Pr)^{-1/5} \]
\[ Nu \sim (Ra*_{H} \ Pr)^{1/5} \]

The validity of these scaling results can be tested by referring to more exact analyses.

Integral analysis

\[ Nu = \frac{2}{360^{1/5}} \left( \frac{Pr}{\frac{4}{5} + Pr} \right)^{1/5} Ra*^{1/5} \]

The similarity solution

\[ Nu = \begin{cases} 0.616Ra*^{1/5} & (Pr \rightarrow \infty) \\ 0.644Ra*^{1/5} Pr^{1/5} & (Pr \rightarrow 0) \end{cases} \]
CONJUGATE BOUNDARY LAYERS

- There are many engineering situations in which the vertical wall that heats a buoyant boundary layer is itself heated on the back side by a sinking boundary layer. Such is the case in walls, partitions, and baffles encountered regularly in the thermal design of living quarters and insulation systems.
- Boundary layers form on both sides of the wall; however, the wall temperature or heat flux are not known a priori.
- The wall temperature “floats” to an equilibrium distribution between the two extreme temperatures maintained by the two reservoirs.
- One of the contributions of the previous analyses is to show that the wall heat flux distribution negotiated between two conjugate natural convection boundary layers is approximated satisfactorily by the $q'' = constant$ model. Therefore, an estimate of the $\text{Nu}(\omega, RaH)$ relationship can be obtained by adding in series the three resistances constituted by the two $q'' = constant$ boundary layers sandwiching the wall.
The solution method consists of writing integral conservation equations for both sides of the wall, with the additional complication that the wall temperature $T_0(y)$ is unknown.

The additional equation necessary for determining $T_0$ is the condition of heat flux continuity in the $x$ direction, from one face of the wall to the other.

The heat transfer rate (hence, the ratio $\frac{Nu_0-H}{Ra^{1/4}H}$) decreases as the wall thickness resistance parameter $\omega$ increases.

$$\omega = \frac{t}{Hk_w} \frac{k}{Ra^{1/4}_H}$$

Wall thermal resistance divided by the thermal resistance of one boundary layer

$$\frac{(H Ra^{1/4}_H)}{Hk}$$
Example

Estimate the $\text{Nu}_{0-H}(\omega, Ra_H)$ function by matching in series the thermal resistance of one $q'' = \text{constant boundary layer}$, the thermal resistance of the wall, and finally, the resistance of another $q'' = \text{constant boundary layer}$. Assume that the temperature along each face of the wall is $y$-independent and equal to the actual temperature averaged over the wall height.

\[ q^H = \frac{T_H - T_C}{R_1 + R_2 + R_3} \]

\[ R_2 = \frac{t}{k_w H} \]

\[ \text{Nu} = \frac{2}{360^{1/5}} \left( \frac{Pr}{4 + Pr} \right)^{1/5} R_{a_y}^{1/5} \]

\[ R_{a_H} = \frac{g \beta H^4 q''}{\alpha v k} \]

\[ \frac{q'' y}{T_H - T_L (y)} = \frac{2}{360^{1/5}} \left( 1 + \frac{0.8}{Pr} \right)^{-1/5} \left( \frac{g \beta q'' y^4}{\alpha v k} \right)^{1/5} \]

\[ T_H - T_L (y) = 1.623 \frac{q''}{k} \left( 1 + \frac{0.8}{Pr} \right)^{1/5} \left( \frac{g \beta q''}{\alpha v k} \right)^{-1/5} y^{1/5} \]
\[ T_H - T_L(y) = 1.623 \frac{q''}{k} \left(1 + \frac{0.8}{Pr}\right)^{1/5} \left(\frac{g\beta q''}{\alpha\nu k}\right)^{-1/5} y^{1/5} \]

Averaging this temperature difference from \( y = 0 \) to \( y = H \) we obtain

\[ \overline{T_H - T_L} = 1.623 \frac{q''}{k} \left(1 + \frac{0.8}{Pr}\right)^{1/5} \left(\frac{g\beta q''}{\alpha\nu k}\right)^{-1/5} \frac{H^{6/5}}{6/5} \frac{1}{H} = q''H \frac{1.352}{k} \left(1 + \frac{0.8}{Pr}\right)^{1/5} Ra_H^{-1/5} \]

\[ q''H = \frac{T_H - T_C}{(2)(1.352) \left(1 + \frac{0.8}{Pr}\right)^{1/5} Ra_H^{-1/5} + \frac{t}{k_w H}} \]

\[ Nu_{0-H} = \frac{q''H}{(T_H - T_C)k}, \quad \omega = \frac{t}{H} \frac{k}{k_w} Ra_H^{1/4}. \]

\[ Nu_{0-H} = \left[2.704 \left(1 + \frac{0.8}{Pr}\right)^{1/5} Ra_H^{-1/5} + \omega Ra_H^{-1/4}\right]^{-1} (*) \]

In view of the relationship between \( Ra_H \) and \( Ra_H \), \( Ra_H = \frac{g\beta H^4 q''}{\alpha\nu k} = Ra_H Nu_{0-H} \)

eq. (*) provides an engineering estimate for the function \( Nu_{0-H} (Ra_H, \omega) \). In the limit \( \omega \to 0 \), this function is

\[ Nu_{0-H} = 0.37 \left(1 + \frac{0.8}{Pr}\right)^{-1/5} Ra_H^{1/5} Nu_{0-H}^{1/5} \]

\[ Nu_{0-H} = 0.288 \left(1 + \frac{0.8}{Pr}\right)^{-1/4} Ra_H^{1/4} \]
VERTICAL CHANNEL FLOW

- If the boundary layer thickness scales are much smaller than the wall-to-wall spacing $D$, the flow along one wall may be regarded as a wall jet unaffected by the presence of another wall.
- If the boundary layer grows to the point that its thickness becomes comparable to $D$, the two wall jets merge into a single buoyant stream rising through the chimney formed by the two walls.
VERTICAL CHANNEL FLOW

momentum equation in y direction

\[ \rho \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial P}{\partial y} + \mu \nabla^2 v - \rho g \]

channel is long enough so that the \( u \) scale becomes sufficiently small

\[ u = 0 \quad \text{and} \quad \frac{\partial v}{\partial y} = 0 \]

momentum equation in x direction implies that:

\[ \frac{\partial P}{\partial y} = \frac{dP}{dy} = -\rho_\infty g \]

\[ \frac{d^2 v}{dx^2} = -\frac{g \beta}{v} (T - T_\infty) \]

The velocity and temperature profiles, are coupled; hence, The momentum and the energy equation must be solved simultaneously.

We can assume that in the fully developed region between two isothermal walls, the temperature difference can be approximated by \( T_0 - T_\infty \). \( (T_0 - T << T_0 - T_\infty) \)
VERTICAL CHANNEL FLOW

\[ v = \frac{g\beta D^2(T_0 - T_\infty)}{8 \nu} \left[ 1 - \left( \frac{x}{D/2} \right)^2 \right] \Rightarrow \dot{m} = \frac{\rho g \beta D^3(T_0 - T_\infty)}{(12) \nu} \]

Total heat transfer rate between stream and channel walls: \( q' = \dot{m}(\text{outlet enthalpy} - \text{inlet enthalpy}) = \dot{m}c_p(T_0 - T_\infty) \)

Average heat flux: \( q''_{0-H} = q'/(2H) \)

Overall Nusselt number:
\[ \frac{q''_{0-H}H}{(T_0 - T_\infty)k} = \frac{Ra_D}{24} \]

\[ Ra_D = \frac{g\beta D^3(T_0 - T_\infty)}{\alpha \nu} \]
consider the heat transfer from a vertical heated wall \((T_0)\) to an isothermal fluid reservoir moving upward \((T_\infty, U_\infty)\), that is, in the same direction as the natural wall jet present when \(U_\infty = 0\).

**The key question:**

Under what conditions is the combined natural–forced phenomenon characterized by the scales of pure natural convection, and conversely, under what conditions is it characterized by the scales of pure forced convection? In other words, what is the criterion for the transition from one convection mechanism to another?

The thermal distance between the heat-exchanging entities

\[
\begin{align*}
(\delta_T)_{\text{NC}} &\sim y \, \text{Ra}_y^{-1/4} & (\text{Pr} > 1) \\
(\delta_T)_{\text{FC}} &\sim y \, \text{Re}_y^{-1/2} \, \text{Pr}^{-1/3} & (\text{Pr} > 1)
\end{align*}
\]
The type of convection mechanism is decided by the smaller of the two distances, \((\delta_T)_{NC}\) or \((\delta_T)_{FC}\), because the wall will leak heat to the nearest heat sink (or because currents seek and construct paths of greater access, or faster mixing).

\[
\begin{align*}
(\delta_T)_{NC} &< (\delta_T)_{FC} & \text{natural convection} & \Pr > 1 \\
(\delta_T)_{NC} &> (\delta_T)_{FC} & \text{forced convection} & 
\end{align*}
\]

The scale criterion for transition from natural to forced convection is

\[
\frac{\text{Ray}^{1/4}}{\text{Re}^{1/2} \text{Pr}^{1/3}} \begin{cases} 
> O(1) & \text{natural convection} \\
< O(1) & \text{forced convection} 
\end{cases} \Pr > 1
\]

\[
\frac{\text{Bo}^{1/4}}{\text{Pe}^{1/2}} \begin{cases} 
> O(1) & \text{natural convection} \\
< O(1) & \text{forced convection} 
\end{cases} \Pr < 1
\]
local similarity solution

- Forced convection dominates at small values of the group $\frac{Gr_y}{Re_y^2}$, while natural convection takes over at large values of the same parameter.
- The knee in each Nusselt number curve shifts to the right as Pr increases: This effect is due to the fact that the abscissa parameter used, $\frac{Gr_y}{Re_y^2}$, is not the same as the dimensionless group that serves as transition parameter
CONVECTIVE HEAT TRANSFER - CHAPTER 5

By: M. Goharkhah

Graph showing the relationship between the Nusselt number divided by the Reynolds number to the power of 1/6 and the Prandtl number. The graph includes curves for different Prandtl numbers (Pr = 100, Pr = 10, Pr → ∞) and highlights the pure natural convection and pure forced convection regimes. The graph is redrawn from the data of Ref. 29 and Fig. 4.13.
• The boundary layer flow remains laminar if \( y \) is small enough so that the Rayleigh number \( Ra_y \) does not exceed a critical value.
• Until recently, it was thought that the transition to turbulent flow occurs at the \( y \) position where \( Ra_y \sim 10^9 \), regardless of the value of the Prandtl number.
• Bejan and Lage, showed that it is the Grashof number of order \( 10^9 \) (i.e., not the Rayleigh number of order \( 10^9 \)) that marks the transition in all fluids:
Vertical Walls

Isothermal-wall correlation

\[ \overline{Nu}_y = \left\{ 0.825 + \frac{0.387 \text{Ra}_y^{1/6}}{\left[ 1 + (0.492/\text{Pr})^{9/16} \right]^{8/27}} \right\}^2 \]

Churchill and Chu

\( 10^{-1} < \text{Ra}_v < 10^{12} \)

\[ \overline{Nu}_y = (0.825 + 0.325 \text{Ra}_y^{1/6})^2 \quad (\text{Pr} = 0.72) \]

In the case of air

In the laminar range, \( \text{Gr}_y < 10^9 \), the following correlation represents the experimental data more accurately

\[ \overline{Nu}_y = 0.68 + \frac{0.67 \text{Ra}_y^{1/4}}{\left[ 1 + (0.492/\text{Pr})^{9/16} \right]^{4/9}} \]

\[ \overline{Nu}_y = 0.68 + 0.515 \text{Ra}_y^{1/4} \quad (\text{Pr} = 0.72) \]
Vertical Walls

Uniformly heated wall

In fluids of the air–water Prandtl number range, the transition to turbulence occurs in the vicinity of \( \text{Ra}_{\ast y} \sim 10^{13} \)

\[
\begin{align*}
\text{Nu}_y &= 0.6 \text{Ra}_{\ast y}^{1/5} \\
\text{Nu}_y &= 0.75 \text{Ra}_{\ast y}^{1/5} & \text{laminar,} \\
\text{Nu}_y &= 0.568 \text{Ra}_{\ast y}^{0.22} & 10^5 < \text{Ra}_{\ast y} < 10^{13} \\
\text{Nu}_y &= 0.645 \text{Ra}_{\ast y}^{0.22} & 10^{13} < \text{Ra}_{\ast y} < 10^{16} \\
\text{Nu}_y &= 0.55 \text{Ra}_{\ast y}^{1/5} & \text{laminar} \\
\text{Nu}_y &= 0.17 \text{Ra}_{\ast y}^{1/4} & \text{turbulent} \\
\end{align*}
\]

Vliet and Liu

A correlation that is valid for all Rayleigh and Prandtl numbers is

\[
\frac{\text{Nu}_y}{\text{Nu}_{\ast y}} = \left\{ 0.825 + \frac{0.387 \text{Ray}^{1/6}}{[1 + (0.437 / \text{Pr})^{9/16}]^{8/27}} \right\}^2 \\
\text{Nu}_y = (0.825 + 0.328 \text{Ray}^{1/6})^2 & \quad (\text{Pr} = 0.72) \\
\text{Nu}_y \approx 0.107 \text{Ray}^{1/3} & \quad (\text{Pr} = 0.72 \text{ and } \text{Ray} > 10^{10})
\]
Inclined Walls

\[-60^\circ < \phi < 60^\circ\]

- In cases \textit{a and d}—heated wall tilted upward and cooled wall tilted downward—the effect of the angle \(\phi\) is to thicken the tail end of the boundary layer and to give the wall jet a tendency to separate from the wall.

- The opposite effect is illustrated in cases \textit{b and c}, where the wall jet is pinched as it flows over the trailing edge.
Inclined Walls

- g in momentum equation is replaced with $g \cos \phi$ in the buoyancy term. The group $g \cos \phi$ is the gravitational acceleration component oriented parallel to the wall.

- The heat transfer rate in the laminar regime along an isothermal wall can be calculated with

$$\overline{Nu}_y = 0.68 + \frac{0.67Ra_y^{1/4}}{[1 + (0.492/Pr)^{9/16}]^{4/9}}$$

$$Ra_y = \frac{g \cos \phi \beta (T_w - T_\infty)y^3}{\alpha \nu}$$

- for laminar flow over a plate with uniform heat flux

$$Nu_y = 0.6Ra_y^{1/5} \quad \text{laminar, }$$

$$\overline{Nu}_y = 0.75Ra_y^{1/5}$$

$$10^5 < Ra_{\theta y} < 10^{13}$$

$$Ra_{\theta y} = \frac{g \cos \phi \beta q''_{w,y}y^4}{\alpha \nu k}$$
turbulent regime

- It was found that the heat transfer measurements are correlated better using $g$ instead of $g \cos \phi$ in the Rayleigh number.

\[
\overline{Nu}_y = \left\{ 0.825 + \frac{0.387Ra_y^{1/6}}{[1 + (0.492/Pr)^{9/16}]^{8/27}} \right\}^2
\]

\[
Nu_y = 0.568Ra_y^{0.22} \quad \text{turbulent,}
\]
\[
Nu_y = 0.645Ra_y^{0.22} \quad 10^{13} < Ra_y < 10^{16}
\]

for isothermal plates

uniform heat flux
Effect of $\phi$ on the location of the laminar–turbulent transition

- The uniform-flux wall is oriented as in cases $a$ and $d$
- The flux Rayleigh numbers tabulated below mark the beginning and the end of the transition region in water experiments ($Pr \sim 6.5$)

<table>
<thead>
<tr>
<th>uniform-flux wall</th>
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<tbody>
<tr>
<td>$\phi$</td>
<td>$Ra_{\phi,y}$</td>
<td></td>
</tr>
<tr>
<td>0°</td>
<td>$5 \times 10^{12} - 10^{14}$</td>
<td></td>
</tr>
<tr>
<td>30°</td>
<td>$3 \times 10^{10} - 10^{12}$</td>
<td></td>
</tr>
<tr>
<td>60°</td>
<td>$6 \times 10^{7} - 6 \times 10^{9}$</td>
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<table>
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<th>isothermal wall</th>
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<tbody>
<tr>
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<td>$Ra_{y}$</td>
<td></td>
</tr>
<tr>
<td>0°</td>
<td>$8.7 \times 10^{8}$</td>
<td></td>
</tr>
<tr>
<td>20°</td>
<td>$2.5 \times 10^{8}$</td>
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</tr>
<tr>
<td>45°</td>
<td>$1.7 \times 10^{7}$</td>
<td></td>
</tr>
<tr>
<td>60°</td>
<td>$7.7 \times 10^{5}$</td>
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</table>